Reg. No. : ....

Name : ....

K22P 3322

IV Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, April 2022
(2018 Admission Onwards)
MATHEMATICS
MAT 4E02 : Fourier and Wavelet Analysis

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each carries 4 Marks.**

1. Define the conjugate reflection of $\omega \in l^2(Z_N)$. For any $z, \omega \in l^2(Z_N)$ and $k \in Z$, prove that $z * \tilde{\omega}(k) = \langle z, R_k\omega\rangle$.

2. Explain downsampling operator and upsampling operator.

3. If $N = 2M$ for some natural number M, $z \in l^2(Z_N)$ and $\omega \in l^2(Z_{N/2})$, then prove that $D(z)*\omega = D(z*U(\omega))$.

4. If $z \in l^2(Z)$ and $\omega \in l^1(Z)$, show that $z* \omega \in l^2(Z)$ and $||z*\omega|| \leq ||\omega||_1||z||$.

5. Define translation-invariant linear transformation on $l^2(Z)$. If $T : l^2(Z) \rightarrow l^2(Z)$ is bounded and translation-invariant, then show that $T(z) = b*z$ for all $z \in l^2(Z)$, where $b = T(\delta)$.

6. If $z \in l^2(Z)$, show that $(z^*)\hat{}(\theta) = z\hat{}(\theta + \pi)$.

7. If $f,g \in L^1(R)$, show that $f*g \in L^1(R)$ and $||f*g||_1 \leq ||f||_1||g||_1$.

8. If $f,g \in L^1(R)$, and if $\hat{f}, \hat{g} \in L^1(R)$, prove that $\langle \hat{f},\hat{g}\rangle = 2\pi\langle f,g\rangle$. **(4x4=16)**

P.T.O.

PART – B

Answer **any four** questions from this part, without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

9. a) Let $w \in l^2(Z_N)$. Then show that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $l^2(Z_N)$ if and only if $|\hat{w}(n)| = 1$ for all $n \in Z_N$.

   b) Suppose M is a natural number, N = 2M and $z \in l^2(Z_N)$. Define $z^* \in l^2(Z_N)$ by $z^*(n) = (-1)^n z(n)$ for all n. Then show that $(z^*)^\wedge(n) = \hat{z}(n + M)$ for all n.

10. Suppose M is a natural number and N = 2M. If $u, v \in l^2(Z_N)$, show that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal basis for $l^2(Z_N)$ if and only if the system matrix A(n) of u, v is unitary for each n = 0, 1, ... , M – 1.

11. Suppose M is a natural number, N = 2M and $u,v,s,t \in l^2(Z_N)$. Show that $\tilde{t} * U(D(z * \tilde{v})) + \tilde{s} * U(D(z * \tilde{u})) = z$ for all $z \in l^2(Z_N)$ if and only if $A(n)\begin{bmatrix} \hat{s}(n) \\ \hat{t}(n) \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$ for each n = 0, 1, ... ,N –1, where A(n) is the system matrix of u, v.

12. If $2^p | N$, explain the construction of a p^{th} stage wavelet basis for $l^2(Z_N)$ from a given p^{th} stage wavelet filter sequence.

Unit – II

13. a) If $\{a_j\}_{j \in Z}$ is an orthonormal set in a Hilbert space H, and if $f \in H$, show that the sequence $\{\langle f, a_j \rangle\}_{j \in Z} \in l^2(Z)$.

    b) Show that an orthonormal set $\{a_j\}_{j \in Z}$ in a Hilbert space H is a complete orthonormal set if and only if $f = \sum_{j \in Z} \langle f, a_j \rangle a_j$ for all f in H.

14. a) Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta} \rangle = 0$ for all $n \in Z$, show that $f(\theta) = 0$ a.e.

    b) If $z \in l^2(Z)$ and $\omega \in l^1(Z)$, prove that $(z*\omega)^\wedge(\theta) = \hat{z}(\theta)\, \hat{w}(\theta)$ a.e.

15. If $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invarient linear transformation, then show that there exists $\lambda_m \in C$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$ for each $m \in Z$.

16. Suppose that $u,v \in l^1(Z)$. Show that $B = \{R_{2k}v\}_{k \in Z} \cup \{R_{2k}u\}_{k \in Z}$ is a complete orthonormal set in $l^2(Z)$ if and only if the system matrix $A(\theta)$ is unitary for all $\theta \in [0, \pi)$.

Unit – III

17. Define approximate identity. Suppose $f \in L^1(R)$ and $\{g_t\}_{t>0}$ is an approximate identity. Then show that for every Lebesgue point x of f, $\lim_{t \to 0^+} g_t * f(x) = f(x)$.

18. Define Fourier transform and inverse Fourier transform on R. Suppose $f \in L^1(R)$ and $\hat{f} \in L^1(R)$, then show that $\frac{1}{2\pi}\int_R \hat{f}(\xi)e^{ix\xi}\,d\xi = f(x)$ a.e. on R. Use this to establish the uniqueness of Fourier transform.

19. Suppose $f \in L^2(R)$, $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions such that $f_n, \hat{f}_n \in L^1(R)$ for each n, and $f_n \to f$ in $L^2(R)$ as $n \to \infty$. Show that $\left\{\hat{f}_n\right\}_{n=1}^{\infty}$ converges to a unique $F \in L^2(R)$. Also, show that if $f \in L^1(R) \cap L^2(R)$, then $F = \hat{f}$.

20. If $f, g \in L^2(R)$, prove that $\langle \hat{f}, \hat{g} \rangle = 2\pi \langle f, g \rangle$ and $\|\hat{f}\| = \sqrt{2\pi}\|f\|$. If $f \in L^2(R)$ and if $\{f_n\}_{n=1}^{\infty}$ is a sequence of L^2 – functions such that $f_n \to f$ in $L^2(R)$ as $n \to \infty$, then prove that $\hat{f}_n \to \hat{f}$ in $L^2(R)$ as $n \to \infty$.

(4×16=64)